ENTANGLEMENT PROBABILITY DISTRIBUTION OF BI-PARTITE RANDOMISED STABILIZER STATES

OSCAR C.O. DAHLSTEN[a] and MARTIN B. PLENIO

Institute for Mathematical Sciences, Imperial College London
London SW7 2PE, UK
and
QOLS, Blackett Laboratory, Imperial College London
London SW7 2BW, UK

We study the entanglement properties of random pure stabilizer states in spin-$\frac{1}{2}$ particles. We obtain a compact and *exact* expression for the probability distribution of the entanglement values across any bipartite cut. This allows for exact derivations of the average entanglement and the degree of concentration of measure around this average. We also give simple bounds on these quantities. We find that for large systems the average entanglement is near maximal and the measure is concentrated around it.

Keywords: Entanglement, Stabilizer States, Probability Distribution, Average, Typical Entanglement.

1 Introduction

Entanglement is a fundamental resource for quantum information processing. The classification and quantification of this resource in two- and many-partite systems is therefore of significant concern, as can be seen in the review papers [1, 2, 3, 4, 5, 6, 7]. Over the past 10 years or so the properties of bi-partite entanglement have been explored in some detail and many of its basic features are now reasonably well understood. However, the entanglement properties of multipartite systems are far more complex and our current understanding of this setting is limited.

There are various approaches that one might take to achieve progress. Firstly, one may impose additional constraints on the set of states and/or the set of operations that one is interested in, ideally without significantly reducing the variety of possible qualitative entanglement structures. In this context an interesting class of states that arises is that of stabilizer states. Despite having various restrictions, these stabilizer states possess a rich entanglement structure exhibiting multi-partite entanglement[8, 9, 10, 11, 12, 13].

With increasing numbers of particles the finest classification of entanglement results in a number of types of entanglement that grows exponentially with the number of particles. However, many of these types may be atypical in that their probability, with respect to some

[a]oscar.dahlsten@imperial.ac.uk

natural measure on the set of states, vanishes in the limit of large numbers of particles. This suggests a different approach that studies only the entanglement properties of typical states. Even before the emergence of quantum information such questions had already been of interest. An example is the study of the expected entropy [14, 15, 16] of a subsystem when averaged over the invariant measure of pure states. In quantum information this entropy has received considerable recent attention as it is precisely the entanglement between this subset and the remainder of the system. The mean entanglement has been studied, as well as the probability for deviations from this mean. These can be shown to be exponentially decreasing with the difference from the mean, a property known as 'concentration of measure' [18].

In the present paper we are interested in a combination of these two approaches, i.e. we study the typical and average entanglement of random stabilizer states. These depend on the probability distribution of entanglement, which gives the probability of finding a certain entanglement in a stabilizer state that has been picked at random. Therefore the objective here is to find and study the entanglement probability distribution of randomised stabilizer states.

The paper is organized as follows. In section 2 we provide some basic results concerning random states, define stabilizer states, the basic technical tools and results that will be used subsequently. In section 3 we state and prove our main result. We study the probability distribution of the entanglement across any given bipartite split of the system. We provide a compact, explicit and exact formula for this probability distribution and present its proof. In section 4 we use this result to study the average entanglement of a set of spins versus the rest and demonstrate that this distribution implies a concentration of measure, i.e. for large numbers of particles the probability that the entanglement of a specific state will deviate from the mean value decreases exponentially with that deviation.

2 Basic techniques and definitions

In the following we present some basic results concerning general random quantum states as well as basic tools for the description of entanglement in stabilizer states.

2.1 Entanglement of randomised quantum systems

Consider a system of N spin-1/2 particles and pick random quantum states from the unitarily invariant distribution. This may for example concern, in an idealized setting, a gas of two-level atoms. As the atoms collide and interact at random their energy levels become entangled. Asymptotically the distribution on pure states becomes uniform, such that any pure state is equally likely. For such a distribution of states we may ask: What is the average entropy of entanglement $\mathbb{E}S_A(N_A, N_B)$ of a set of N_A spins? Some of the first studies of this question were [14, 17] and the explicit solution('Page's conjecture') was conjectured in [15] and proven in [16]. The explicit solution is given by

$$\mathbb{E}S_A(N_A, N_B) = \frac{1}{ln2}\left(\sum_{k=2^{N_B}+1}^{2^{N_A+N_B}} \frac{1}{k} - \frac{2^{N_A}-1}{2^{N_B+1}}\right) \tag{1}$$

with the convention that $N_A \leq N_B$ and where $N_A + N_B = N$, the total number of particles.

This can be used to show that the average entanglement is very nearly maximal, meaning close to N_A, for large quantum systems, i.e. $N \gg 1$. Hence one concludes that a randomly

chosen state will be nearly maximally entangled with a large probability. Indeed, it was recently shown that the probability that a randomly chosen state will have an entanglement E that deviates by more than δ from the mean value $\mathbb{E}S_A(N_A, N_B)$ decreases exponentially with δ^2. This phenomenon, alongside many other properties, which is known as concentration of measure of entanglement, was proven and extensively studied in [18].

2.2 Entanglement of stabilizer states

As it is our aim to study the typical properties of stabilizer states we use this subsection to present a number of basic tools and observations that are useful in this context.

Stabilizer states are a discrete subset of general quantum states, which can be described by a number of parameters scaling polynomially with the number of qubits in the state [8, 9, 22].

A *stabilizer operator* on N qubits is a tensor product of operators taken from the set of Pauli operators

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and the identity I. An example for $N = 3$ would be the operator $g = X \otimes I \otimes Z$. A set $G = \{g_1, \ldots, g_K\}$ of K mutually commuting stabiliser operators that are independent, i.e. $\prod_{i=1}^{K} g_i^{s_i} = I$ exactly if all s_i are even, is called a *generator set*. For $K = N$ a generator set G uniquely determines a single state $|\psi\rangle$ that satisfies $g_k|\psi\rangle = |\psi\rangle$ for all $k = 1, \ldots, N$. Such a generating set generates the stabilizer group. Each unique such group in turn defines a unique *stabilizer state*.

A first observation that will be useful for the following considerations is the fact that the bipartite entanglement of a stabilizer state, i.e. the entanglement across any bipartite split, takes only integer values [10, 12].

In the proof of our main theorem in section III below we will furthermore use results from [12]. In particular we use their result 1, that a stabilizer state of E ebits can be generated from:

- 'local generators' of type $g_A \otimes I_B$ and $I_A \otimes g_B$ where g refers to a member of the Pauli group.
- 'non-local generators' of type $g_A \otimes g_B$. These generators come in pairs where the entries corresponding to Alice (and Bob respectively) anti-commute.

The subgroups generated by the local elements are labelled $\mathcal{S}_A$ and $\mathcal{S}_B$ respectively, and the subgroup generated by the non-local elements $\mathcal{S}_{AB}$. The entanglement E of a state is given by the number of pairs in the minimal generator set of all non-local pairs, i.e.

$$E = |\mathcal{S}_{AB}|/2. \tag{3}$$

For example the GHZ state $|000\rangle + |111\rangle$ with respect to the division $N_A = 2$ (first two particles) and $N_B = 1$ is defined by the generator set $\langle XXX, IZZ, ZZI\rangle$. The local generators on Alice's and Bob's part respectively are given by $\langle ZZI\rangle$ and $\langle I\rangle$. The non-local part is $\langle XXX, IZZ\rangle$ which consists of only one pair so that the entanglement is $E = 1$. We will also use Eq. 6 from [12] giving entanglement E as

$$E = N_A - |S_A| = N_B - |S_B| \tag{4}$$

where the |.| refer to the size of the minimal generator set of the subgroup in question.

Finally we use the fact that in order for the stabilizer state to be non-trivial it is necessary and sufficient that the elements of the stabilizer group (a) commute, and (b) are not equal to -I [22].

3 Main Result- Entanglement Probability Distribution

Let us consider the setting shown in fig. 1 where a system of N spin-$\frac{1}{2}$ particles is split into two arbitrarily sized subsets consisting of N_A and N_B particles such that $N = N_A + N_B$. Consider now a randomly chosen pure stabilizer state on the N particles which is chosen according to the uniform measure. Generally the spins in set A will exhibit entanglement with the spins

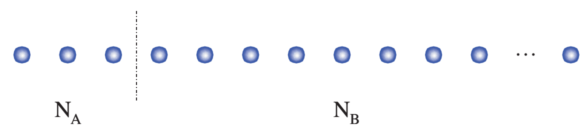

Fig. 1. The spins are grouped into two sets A and B. Set A contains N_A spins, set B contains N_B and the total number of spins is $N = N_A + N_B$.

in set B and this entanglement will take discrete values $E \in \{0, 1, 2, \ldots, min(N_A, N_B)\}$. The main result of this paper is an exact and compact expression for the probability distribution of the entanglement values E under randomised stabilizer states.

Theorem I −*Entanglement probability distribution:* In a system of N spins where $N_A(N_B)$ is Alice's (Bob's) number of qubits the probability of finding that entanglement between A and B equals E in a randomly chosen pure stabilizer state is given by

$$P(E) = \frac{\prod_{i=1}^{N_A}(2^i + 1)}{\prod_{k=N-N_A+1}^{N}(2^k + 1)} \prod_{j=1}^{E} \frac{\left(2^{N-N_A+1-j} - 1\right)\left(2^{N_A+j} - 2^{2j-1}\right)}{2^{2j} - 1}$$

where E is an integer and $0 \leq E \leq min(N_A, N_B)$.

Proof: The strategy of the proof is to firstly define the probability distribution on stabilizer states. Then we count the number of states for which the entanglement between sets A and B is given by E and label this number by n_E. Finally the probability weight is multiplied by n_E to obtain the probability distribution of entanglement.

For a set of N spins we denote the *total number of possible stabilizer states* by $n_{tot}(N)$. Randomising these states is then achieved by the random application of unitary maps that take any stabilizer states to another stabilizer state. The probability distribution that is invariant under randomisation is the uniform distribution. Therefore the probability of picking a stabilizer state j at random is given by $p_j = \frac{1}{n_{tot}(N)}$.

For given values of N, N_A and N_B, let E denote the amount of entanglement between set A and B, and let $n_E^{(N_A)}(N)$ denote the number of stabilizer states on N particles realizing this value of entanglement. It will be our task to determine $n_E^{(N_A)}(N)$ exactly. It is clear that for any N_A it must be the case that $n_{tot}(N)$ satisfies:

$$n_{tot}(N) = \sum_{E=0}^{\lfloor N/2 \rfloor} n_E^{(N_A)}(N) \tag{5}$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than x.

The proof of our theorem will now be presented in the form of several useful lemmas.

Lemma I −*Total number of states:* The total number of distinct stabilizer states for N particles is given by

$$n_{tot}(N) = 2^N \prod_{k=1}^{N} (2^k + 1) \,. \tag{6}$$

Proof: This is proven in [11] and an alternative proof employing different techniques is found in [20]□

Lemma II −*Number of separable states:* The number of pure stabilizer states for which the spins of sets A and B are *not* entangled is given by

$$n_0^{(N_A)}(N) = n_{tot}(N_A) n_{tot}(N_B). \tag{7}$$

Proof: A pure state does not exhibit entanglement between the spins of sets A and B exactly if it is a tensor product between the two sets. Therefore the total number of unentangled pure stabilizer states is the product of the total number of states on each part□

Lemma III −*Invariant ratio of probability distribution:* For $0 \leq E \leq min(N_A, N_B)$ and $N = N_A + N_B$ we have

$$\frac{n_E^{(N_A+1)}(N+1)}{n_E^{(N_A)}(N)} = \frac{n_E^{(N_A+1)}(N_A+1+E)}{n_E^{(N_A)}(N_A+E)} \,. \tag{8}$$

Proof: This is true exactly if there is, for arbitrary N_A and $0 \leq E \leq min(N_A, N_B)$, a constant ratio

$$\frac{n_E^{(N_A+1)}(N+1)}{n_E^{(N_A)}(N)} = f(E, N_A) \tag{9}$$

where $f(E, N_A)$ is some function that only depends on E and N_A but *not* on N_B. To see this we now employ the methods described in section II. Note firstly that the numbers n_E are defined by counting how many stabilizer states there are corresponding to a given entanglement. This is given by the number of distinct stabilizer groups with that entanglement. For this number to be non-zero we only consider counting it when $0 \leq E \leq min(N_A, N_B)$. The stabilizer groups have to be Abelian and contain no negative identity, as mentioned in the introduction. Consider now the number of ways of picking stabilizer group elements, under these restrictions, on Bob's side. We can call this number β and the corresponding number on Alice's side α. The problem is simplified by counting the number of distinct generator sets of the group. Several generator sets will generate a given group, so the number of generator sets is larger than the number of groups by some overcount factor. We now show that β, as well as the overcount factor, are invariant during the transition $\tau : N_A \mapsto N_A + 1$ under constant entanglement E.

Firstly note that the contribution of Bob's side to β, and to the overcount factor, is independent of $|S_A|$. This is because there are only identities in the corresponding terms on

Bob's side. It does depend on $|S_B|$ and $|S_{AB}|$ though. Eq. 4 implies that $|S_A|$, but not $|S_B|$ changes under the transition $\tau : N_A \mapsto N_A + 1$. The condition of constant E is equivalent to constant $|S_{AB}|$ by Eq. 3.

So $\tau : \beta \mapsto \beta$ and $\tau : \alpha \mapsto \alpha'$. Therefore Bob's contribution to the ratio (9) cancels; the ratio is independent of N_B. □

Now we note that the following closed form expressions

$$n_E^{(N_A)}(N) = \left(2^{N+1-N_A-E} - 1\right) \frac{2^{N_A+E} - 2^{2E-1}}{2^{2E} - 1} n_{E-1}^{(N_A)}(N) \tag{10}$$

satisfy equations (5), (6), (7) and (8), for $0 \leq E \leq min(N_A, N_B)$, i.e. for all possible settings. What remains to be shown is that this choice is indeed unique, i.e. that we can construct $n_E^{(N_A)}(N)$ recursively and uniquely from equations (5), (6), (7) and (8). We proceed in the following steps.

- Note that we know $n_E^{(N_A)}(N)$ for $N_A = 1$ and all valid choices of N for $E = 0$ by virtue of Lemma II and then for $E = 1$ by virtue of eq. (5).
- Assume that we have obtained $n_E^{(N_A)}(N)$ for $N_A = 1, 2, \ldots, r_0$ and all valid choices of $E \in \{0, \ldots, N_A\}$ and $N \in \{N_A + E, \ldots\}$. Now we will demonstrate that this uniquely defines $n_E^{(r_0+1)}(N)$ for all $E \in \{0, 1, \ldots, r_0 + 1\}$ and all $N \in \{r_0 + 1 + E, \ldots\}$.
    - To this end realize first that $n_{E=0}^{(r_0+1)}(N)$ is again known for all N by virtue of Lemmas I and II. Now assume that for fixed $r_0 + 1$ we have found $n_x^{(r_0+1)}(N)$ for $x \in \{0, 1, \ldots, E\}$ and all N.
    - We find $n_{E+1}^{(r_0+1)}(N)$ for all N in the following way. First we use eq.(5) to obtain $n_{E+1}^{(r_0+1)}(E+r_0+2) = n_{tot}(E+r_0+2) - \sum_{j=0}^{E} n_j^{(r_0+1)}(E+r_0+2)$. Now we employ the recursion relation in Lemma III to obtain $n_{E+1}^{(r_0+1)}(N+1) = \frac{n_{E+1}^{(r_0+1)}(E+r_0+2)}{n_{E+1}^{(r_0)}(E+r_0+1)} n_{E+1}^{(r_0)}(N)$ where each term on the right hand side is already known by assumption.

  This completes the construction for arbitrary N which in turn completes the construction for arbitrary E. Therefore, all $n_E^{(N_A)}(N)$ are uniquely determined. It is now cumbersome but straightforward to check that eq.(10) satisfies all the recursion relations which in turn implies that it is the unique solution presenting the correct values for $n_E^{(N_A)}(N)$.

This finishes the proof for Theorem I □

4 Implications

This section will present a discussion of the implications of Theorem I.

4.1 Product-free form

Theorem I can be more conveniently evaluated in the format free from products
Corollary I −*Product-free form:* The probability of E entanglement is

$$P(E) = 2^{\frac{(N_A - N_B)^2}{4} - (N/2-E)^2 + \Sigma_1 + \Sigma_2} \tag{11}$$

where N_A, N_B and N are as defined earlier, and

$$\Sigma_1 = \sum_{j_1=1}^{N_A} log_2(1+2^{-j_1}) + \sum_{j_2=1}^{N-N_A} log_2(1+2^{-j_2}) - \sum_{j_3=1}^{N} log_2(1+2^{-j_3}) \tag{12}$$

and finally

$$\Sigma_2 = \sum_{k=1}^{E} log_2(1-2^{k-1-N_A}) + log_2(1-2^{-N+N_A-1+k}) - log_2(1-2^{-2k})\,. \tag{13}$$

Proof: This is a reexpression of Theorem I□

We will see Σ_1 and Σ_2 converge to numbers of magnitude $\ll N$ when N is large so they can be seen as minor modulations to the leading Gaussian-type behaviour. This statement will be made more rigorous later on in this section.

4.2 Comments on entanglement probability distribution

It is instructive to consider qualitatively what kind of distributions Theorem I describes, before making precise mathematical arguments. Firstly consider the simplest non-trivial case, that of N=2, $N_A = N_B = 1$. Here one obtains that the number of states with one ebit is $n_1 = 24$, whereas $n_0 = 36$ with $n_{tot} = 60$. Hence the expected entanglement is 0.4. If we now increase Bob's size to $N_B = 2$, the formulae show that $n_0 = 360$, $n_1 = 720$ with $n_{tot} = 1080$, giving an average entanglement of $\frac{720}{1080} = \frac{2}{3}$. When considering all possible N_A under increasing N, one observes that the maximal and near maximal values of entanglement become the main contributors to the number of possible states. This can be seen from Eq. (11) in which the leading terms inside the sum correspond to a Gaussian centred on $N/2$. Note however that although the corrections to this Gaussian, such as the Σ_2 term are small, they contain an E-dependence that adds some subtleties to this behaviour that will require a more rigorous analysis. This can be understood by noting from Eq. (10), that

$$\frac{n_{N_A-1}}{n_{N_A}} = \frac{1}{2^{N-2N_A} - \frac{1}{2}} - \frac{1}{2^N - 2^{-2N_A-1}} \tag{14}$$

where the second term quickly disappears with increasing N. If N_A is constant, the ratio tends to 0 exponentially fast. However if $N_A = \frac{N}{2}$ then $n_{N_{A-1}}/n_{N_A} \to 2$ for large N which implies in fact that the maximum of the probability distribution is shifted to $N/2-1$. In that case one sees from Eq. (10) that $n_{N_{A-2}}/n_{N_{A-1}} \to \frac{2}{9}$, after which the ratios are very small. Hence for this case $N_A - 1$ is the most likely entanglement of a state picked at random rather than N_A.

We can now use the entanglement probability distribution to derive the average entanglement of randomised stabilizer states.

4.3 Average entanglement of randomised stabilizer states

Corollary II –*Average entanglement:* The average entanglement, $\mathbb{E}S_A$, in stabilizer states sampled at random is given by

$$\mathbb{E}S_A = \frac{\prod_{i=1}^{N_A}(2^i+1)}{\prod_{k=N-N_A+1}^{N}(2^k+1)} \sum_{E=1}^{N_A} E \prod_{j=1}^{E} \frac{\left(2^{N-N_A+1-j}-1\right)\left(2^{N_A+j}-2^{2j-1}\right)}{2^{2j}-1}$$

where N_A is the number of qubits belonging to Alice and N is the total number of qubits. We follow, without loss of generality, the convention that $N_A \leq N - N_A = N_B$. The total state is bipartite and pure and the average is taken over a flat distribution on stabilizers.

Proof: We take as a starting point

$$\mathbb{E}S_A = \sum_{E=1}^{N_A} P(E)E \tag{15}$$

which is then evaluated using Theorem I□

It is also worth mentioning that the case $N_A = 1$ yields the simple form

$$\mathbb{E}S_A = \frac{n_{tot} - n_0}{n_{tot}} = 1 - \frac{3}{2^N + 1}. \tag{16}$$

4.4 Concentration of measure

If some property becomes very likely in certain circumstances, we say the measure is becoming concentrated around states with this property. It is known[18] that the measure on general pure states is concentrated around states with near maximal entanglement. Here we show that a similar concentration takes place on stabilizer states, with some subtleties. We know that the mean is close to maximal which we can use as an upper bound. Below we give an exact expression for the probability of picking a state with an entanglement less than the average by some number. We then give an inequality for the same quantity.

Corollary III –*Concentration of measure around mean in stabilizer states:* The probability of picking a stabilizer state such that it is at least ε less than the average $\mathbb{E}S_A$ is given by

$$P(S_A < \mathbb{E}S_A - \varepsilon) = \sum_{E=1}^{\lfloor \mathbb{E}S_A - \varepsilon \rfloor} P(E) \tag{17}$$

where P(E) is given by Theorem I and $\mathbb{E}S_A$ by Corollary II. ε is a constant which can take any value in the range of 1 to $N_A - 1$.

The concentration of measure can also be given in the shape of an inequality by exploiting the fact that the leading behaviour of the probability distribution of entanglement is Gaussian:

$$\sum_{E=1}^{\lfloor \mathbb{E}S_A - \varepsilon \rfloor} P(E) \leq \gamma(N, N_A) \int_1^{1+\lfloor \mathbb{E}S_A - \varepsilon \rfloor} dE e^{-ln2(N/2-E)^2} \tag{18}$$

where

$$\gamma(N, N_A) = 2^{\frac{(N_A - N_B)^2}{4} + s_1 + s_2}\, ; \tag{19}$$

s_1 is defined in Eq. (22). s_2 is given by

$$s_2 = \frac{h^{N_A} + h^{N_B}}{1 - h} - log_2(15/16) + h \tag{20}$$

where for typographical reasons $-log_2(3/4) = h$.

Proof: The terms inside the integral are taken from the product free form of the probability distribution. We then factor out γ. This is possible by using E-independent bounds

on Σ_1 and Σ_2. The latter is an E-independent version of Eq. 26 below. The integral can be taken as a bound to the sum, as the remaining Gaussian is centred on $N/2$, and without loss of generality $N_A \leq N/2$, meaning there is a monotonic rise within the bounds of the integral $\square$

The Gaussian integral can be evaluated using the error function.

4.5 Comparison with general random states

The typical entanglement of stabilizer states is very similar to that of general states. By this statement we refer to two properties of the entanglement in random large general states picked from the unitarily invariant distribution: the average entanglement is near maximal, N_A, and the probability distribution is exponentially concentrated around this average.

The concentration of measure section above shows that the latter is true also for random large stabilizer states. The typical entanglement of stabilizer states is nearly maximal, as is the case for general states. The averages are slightly different though, and the concentration of measure around the average appears to be a bit less abrupt in the stabilizer state case.

Figure 2 shows how the averages of randomised general states and stabilizer states compare for the full range of possible Alice and Bob divisions where the total system is ten qubits.

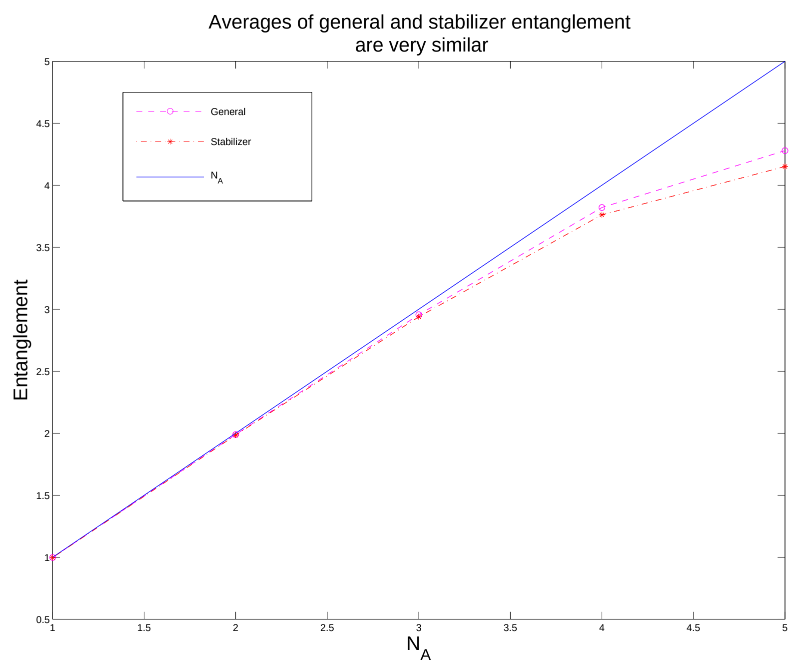

Fig. 2. For a system of ten qubits with N_A belonging to Alice the average entanglement of both general[15] and stabilizer states(Corollary II) are shown.

In order to prove this similarity for arbitrary N_A a good starting point is a rearrangement of Corollary II.

$$\mathbb{E}S_A = N_A(1 - \frac{n_0}{n_{tot}}) - \sum_{E=1}^{N_A - 1} (N_A - E) 2^{\frac{(N_A - N_B)^2}{4} - (\frac{N}{2} - E)^2 + \Sigma_1 + \Sigma_2} \tag{21}$$

which uses the product free form of Theorem I as in Eq. (11). To obtain a lower bound on this we will need the following upper bounds on Σ_1 and Σ_2.

• Σ_1 *upper bound:*

A compact expression for $N_A = 1$ was given in eq. (16) so that we may concentrate on the case $N_A \geq 2$. Then $\Sigma_1 \leq s_1$ where

$$s_1 \equiv log_2(3/2) + log_2(5/4) + \frac{13}{36}\left[1 - \left(\frac{26}{50}\right)^{N_A - 2}\right] . \tag{22}$$

Proof: Firstly note

$$\begin{aligned}
\Sigma_1 &= \sum_{j1=1}^{N_A} log_2\left(1 + 2^{-j1}\right) - \sum_{j3=N_B+1}^{N} log_2\left(1 + 2^{-j3}\right) && (23)\\
&\leq \sum_{j1=1}^{N_A} log_2\left(1 + 2^{-j1}\right) && (24)\\
&= log_2(3/2) + log_2(5/4) + \sum_{j1=3}^{N_A} log_2\left(1 + 2^{-j1}\right) . && (25)
\end{aligned}$$

Now consider using an upper bound of the type $exp(a + bk)$ on $log_2(1 + 2^{-2-k})$, where a and b are constants to be determined. Choose a and b such that the bound is exact for $k = 1$ and $k = 2$ respectively. This yields $a = 2ln(log_2(9/8)) - ln(log_2(17/16)) \cong -1.10825...$ and $b = ln(log_2(17/16)) - ln(log_2(9/8)) \cong -0.664143....$ That $exp(a + bk)$ is indeed an upper bound to $log_2(1 + 2^{-1-k})$ for $k \geq 2$ follows by immediately comparing the gradients of the two functions. One now substitutes this bound into Eq. (25) and uses the standard formula for sums of geometric sequences, as well as bounding the messy-looking logarithm ratios by rational numbers, to recover Eq. (22)□

The upper bound on Σ_2 can be derived employing very similar ideas.

• Σ_2 *upper bound:* To achieve a more compact notation let $h = -log_2(3/4)$

$$\Sigma_2 \leq \frac{1 - h^{-E}}{1 - h}\left[h^{N_A} + h^{N_B}\right] + \frac{4}{3}\left(2^{-2E} - 1\right) log_2\left(15/16\right) + h\,. \tag{26}$$

Proof: Essentially the same method is used as in the bound for Σ_1. We use two inequalities that are valid within the range used only. Firstly

$$-\,log_2\left(1 - 2^{-2j-2}\right) \leq e^{ln(-log_2(15/16)) - 2ln2 + 2ln2j} \tag{27}$$

which was found using an $\exp(a + bj)$ type bound which we required to be exact for $j = 1$. The maximum allowable gradient such that it is an upper bound for $j \geq 1$ was used. Secondly

$$log_2\left(1 - 2^{-x}\right) \leq -e^{ln(-log_2(3/4))(1-x)} \tag{28}$$

which was found using an $\exp(a + bx)$ type bond, where a and b are chosen such that the bound is exact for $x = 1$ and $x = 2$. The gradients of the two functions are such that it will be an upper bound for all points in the range. These two bounds are then inserted into the sum in the definition of Σ_2 and with the standard formula for sums of geometric sequences one recovers Eq. (26)□

Using these bounds in Eq. (21) provides a very tight fit to the exact result.

5 Summary and Outlook

In this paper we have studied the entanglement properties of randomised stabilizer states on N particles. We considered a bi-partite setting with two sets containing N_A and N_B spins with $N = N_A + N_B$ and the entanglement between them. We obtain an exact and compact probability distribution for the possible values of entanglement (which are integers) and use this to give exact values for the average entanglement and the concentration of measure around it with increasing size of the system. It was found that, for large systems, the average entanglement is nearly maximal and the probability of a state having a significantly different entanglement than this average decreases exponentially with the difference. We also compared these results to the case of general random quantum states and found a close similarity.

This suggests that it would be interesting to conduct research into how far this similarity carries, in particular in the context of multi-partite entanglement between large and/or non-contiguous sets of particles and of dynamical features of quantum states under the action of random quantum gates (stabilizer or general unitary gates) [17, 21] and their approach to the equilibrium distribution. These topics will be the subject of a forthcoming publication currently in preparation.

Acknowledgements

We are grateful to K. Audenaert, J. Eisert, D. Gross, J. Oppenheim, T. Rudolph, G. Smith for discussions and especially to S.Virmani as well as an unknown referee for carefully reading the manuscript. This work is part of the QIP-IRC supported by EPSRC (GR/S82176/0) and was supported by The Leverhulme Trust, The Royal Society and the European Union Integrated Project QAP.

Note added: After completion of this research we became aware of work by Graeme Smith and Debbie Leung, [19], which considers the same issues and some additional questions but does not provide exact expressions for the probability distribution for the entanglement of a set of spins versus the rest as we do here.

References

1. M.B. Plenio and V. Vedral (1998), *Teleportation, Entanglement and Thermodynamics in the Quantum World*, Contemp. Phys. **39**, 431.
2. J. Eisert and M.B. Plenio (2003), *Introduction to basics of Entanglement Theory in Continuous Variable Systems* , Int. J. Quant. Inf. **1**, 479.
3. M.B. Plenio and S. Virmani (2005), *An Introduction to Entanglement Measures* , E-print arxiv quant-ph/0504163.
4. J. Eisert and D. Gross (2005), *Multi-Particle Entanglement* , E-print arxiv quant-ph/0505149.
5. M. Horodecki (2001), *Entanglement measures*, Quantum Information and Quantum Computation, **1,1** (pp3-26)
6. W. Wootters (2001), *Entanglement of formation and concurrence*, Quantum Information and Quantum Computation, **1,1** (pp27-44)
7. P. Horodecki, R. Horodecki (2001), *Distillation and bound entanglement*, Quantum Information and Quantum Computation, **1,1** (pp45-75)
8. D. Gottesman, *Stabilizer Codes and Quantum Error Correction*, Caltech PhD thesis.

9. D. Gottesman (1998), *The Heisenberg Representation of Quantum Computers*, E-print arxiv quant-ph/9807006.
10. K.M.R. Audenaert and M.B. Plenio (2005), *Entanglement on mixed stabiliser states: Normal Forms and Reduction Procedures*, New J. Phys. **7**, 170. Note that the Matlab codes in this work can be downloaded from www.imperial.ac.uk/quantuminformation.
11. D.Gottesman and S.Aaronson (2004), *Improved Simulation of Stabilizer Circuits*, Phys. Rev.A, 70:052328.
12. D. Fattal, T. S. Cubitt, Y. Yamamoto, S. Bravyi and I.L. Chuang (2004), *Entanglement in the stabilizer formalism*, E-print arxiv quant-ph/0406168.
13. M. Hein, J. Eisert and H.J. Briegel (2004), *Multiparty Entanglement in Graph States*, Phys. Rev. A **69**, 062311.
14. E.Lubkin (1978), *Entropy of an n-system from its correlation with a k-reservoir*, J. Math. Phys **19**(5), pp.1028-1031 .
15. D.N.Page (1993), *Average entropy of a subsystem.* Phys. Rev. Lett. **71** No.9, (1993).
16. S.K Foong and S.Kanno (1994), *Proof of Page's conjecture on the average entropy of a subsystem.* Phys. Rev. Lett. **72** No. 8, pp.1148-1151.
17. S.Lloyd and H.Pagels (1988), *Complexity as Thermodynamic Depth.* Ann. of Phys. **188**, 186-213.
18. P.Hayden, D.W.Leung and A.Winter (2004), *Aspects of Generic Entanglement*, Comm. Math. Phys. March 2006.
19. G.Smith and D. Leung (2005), *Typical Entanglement of Stabilizer States*, E-print arxiv quant-ph/0510232.
20. D. Gross (2006), *Hudson's Theorem for finite-dimensional quantum systems*, E-print arxiv quant-ph/0602001.
21. J. Calsamiglia, L. Hartmann, W. Dür, H.-J. Briegel (2005), *Spin Gases: Quantum Entanglement Driven By Classical Kinematics*, Phys. Rev. Lett. **95**, 180503.
22. M.A. Nielsen and I.L. Chuang, *Quantum computation and Quantum Information*, Cambridge Univ. Press, Cambridge, UK.